Mice meet worms: topological semantics for the logic of correct submodels

Scott analysis. Foundational theories and related topics

grigorii stepanov

TU Vienna

April 20th, 2026

Intro

Plan

- logics GL.3 and GLP.3;
- topological semantics for provability logic;
- topologies on ordinals;
- some completeness results;

The logic GL

Logic GL is the smallest set of formulæ in $\mathcal{L}_\Box$ closed under modus ponens, that contains classical tautologies and modal axioms which reflect provability nature of the Boxes:

1. $\Box(p \to q) \to (\Box p \to \Box q)$ (Normality)
2. $\Box(\Box p \to p) \to \Box p$ (Löb)

Arithmetical completeness

Fix some gödelian theory T and some valuation function

$$v : \mathrm{var} \to \mathcal{L}_{PA}$$

this can yield the arithmetical interpretation:

$$[\![p]\!]_T = v(p) \qquad [\![\varphi \wedge \psi]\!]_T = [\![\varphi]\!]_T \wedge [\![\psi]\!]_T$$

$$[\![\neg\varphi]\!]_T = \neg[\![\varphi]\!]_T \quad [\![\Box\varphi]\!]_T = \mathsf{Pr}_T(\overline{\ulcorner [\![\varphi]\!]_T \urcorner})$$

$$Log(T) = \{\varphi \in \mathcal{L} : \forall (v : \mathrm{var} \to \mathcal{L}_{PA}) T \vdash [\![\varphi]\!]_T\}$$

Solovay showed that $Log(\mathrm{PA}) = \mathrm{GL}$[1].

[1]Solovay, "Provability interpretations of modal logic".

The logic GL.3

In the same paper Solovay introduced the logic $\mathsf{GL.3} = \mathsf{GL} + .3$, where

$$.3 = \Box(\Box p \to q) \vee \Box(\Box q \wedge q \to p).$$

And established completeness for the interpretation $[\![\Box\varphi]\!] =$ "$[\![\varphi]\!]$ holds in every transitive model of ZFC"[2].

[2] Under assumption that there is at least one uncountable model of ZFC.

The logic GLP(.3)

Aguilera and Pakhomov[3] introduced the logic GLP.3, where GLP is the logic:

1. $[n](p \to q) \to ([n]p \to [n]q)$ (Normality);
2. $[n]([n]p \to p) \to [n]p$ (Löb);
3. $[m]p \to [n][m]p, m \leq n$;
4. $\langle m \rangle p \to [n]\langle m \rangle p, m < n$;
5. $[m]p \to [n]p, m \leq n$;

And GLP.3 $=$ GLP $+$ .3, where:

$$.3 = [n]([n]p \to q) \vee [n]([n]q \wedge q \to p)$$

And established under $V = L$ completeness for the interpretation $[\![[n]\varphi]\!] =$ "$[\![\varphi]\!]$ holds in every Σ_n-correct transitive model of ZFC$''$.

[3] Aguilera and Pakhomov, "The Logic of Correct Models".

Kripke semantics

Let (X, R, v) be such that X is a set, $R \subset X \times X$ is a relation on X and $v : \mathrm{var} \to P(X)$ is a valuation, stating which variables hold at a point $x \in X$. Then one can extend v to an arbitrary modal formula by:

- $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$;
- $v(\neg\varphi) = X \setminus v(\varphi)$;
- $v(\Box\varphi) = \{x \in X : \forall y(xRy \to y \in v(\varphi))\}$;

We say that φ holds at $x \in X$ and write $X, x \Vdash_v \varphi$ if $x \in v(\varphi)$. We say that the model (X, R, v) satisfies φ and write $X \models_v \varphi$ if $v(\varphi) = X$ and we say that the frame (X, R) satisfies φ if (X, R, v) satisfies φ for any v, and write $X \models \varphi$ in such case.

Kripke completeness

We say that a logic is Kripke complete with respect to a class of frames, if it is valid in every frame of this class; and if any non-theorem can be found refuted in some model on a frame of this class.

- GL is Kripke complete with respect to finite irreflexive trees;
- GL.3 is complete with respect to finite strict linear orders;
- yet no Kripke completeness of GLP or GLP.3 is possible;
- neither strong Kripke completeness[4] for GL and GL.3;

[4] i.e. existence of a model that satisfies a consistent set of formulas;

Topological semantics

Topological approach can mitigate the above-mentioned problems. Moreover, natural topological models for natural logics are always fascinating.

Definition

Let (X, τ) be a topological space and $v : \mathrm{var} \to P(X)$ be a valuation. Then one can extend v as follows:

- booleans are as usual;
- $[\![\Diamond\varphi]\!] = d[\![\varphi]\!]$;

Where $\Diamond\varphi = \neg\Box\neg\varphi$ and $dA = \{x \in X : \forall U \in \tau(x \in U \to \exists(y \neq x)y \in U \cap A)\}$ is the set of limit point of A.

Known completeness results

GL is know to be complete with respect to scattered spaces, i.e. spaces (X, τ), where every $A \subset X$ has an isolated point. In such spaces the following is well-defined $\rho(x) = \min\{\alpha : x \notin d^{\alpha+1}X\}$, which corresponds to the well-foundedness forced by the Löb's axiom, which, in turn, corresponds to the induction in PA.
A classical example of a scattered space is an ordinal α take with the interval topology τ_ι generated by sets of the form (β, γ), where $\beta, \gamma < \alpha$.
Moreover, we have:

Theorem (Abashidze, Blass)

GL is complete with respect to $(\omega^\omega, \tau_\iota)$.

Known completeness results

The topological approach yields results unattainable with Kripke semantics.

Theorem (Aguilera, Fernandez-Duque)

GL *is strongly complete with respect to* ω*-bouquets and with respect to* $(\omega^\omega + 1, \tau_\iota)$.

Where ω-bouquets are topological spaces similar to Kripke frames with a different valuation on the nodes with infinitely many successors (see the blackboard).

Theorem (Beklemishev-Gabelaia)

GLP *is complete with respect to ordinals with Beklemishev-Gabelaia topology.*

Completeness results for GL.3

Topologies/filter on ordinals

Given a topology (X, τ) and a point $x \in X$, one can think of the set of the neighbourhoods of x as a filter, called the neighbourhood filter. It is convenient to keep this in mind, when talking about topologies on ordinals. Here are some prominent examples:

- the order topology corresponds to the cobounded filters on the limit ordinals;
- the club topology corresponds to the filter of closed unbounded subsets of ordinals of cofinality $\geq \omega_1$;
- the normal topology corresponds to the filter given by sets, who have measure one with respect to every normal complete ultrafilter on a measurable cardinal;

Club topology

Definition

Let α be an ordinal of uncountable cofinality. We say that $A \subset \alpha$ is a club (closed unbounded subset) if it is:

- unbounded, i.e. for any $\beta < \alpha$ there is $\gamma \in A$ such that $\gamma > \beta$;
- closed, i.e. whenever $\{\alpha_i : i < \delta\} \subset A$ for some $\delta < \mathrm{cof}\alpha$, $\sup_i \alpha_i \in A$;

It is known that clubs generate a filter called the club filter, thus the topology generated by closed sets of ordinals corresponds to the club filter.

Fact (Blass)

It is consistent with ZFC *that:*

- GL *is the logic of the club topology ($V = L$ or $\square_\kappa$ for all $\kappa < \aleph_\omega$);*
- GL *in not the logic of the club topology (Harrington-Shelah model);*

The Axiom of Determinacy

The following definiton is from[5]:

With each subset A of ω^ω we associate the following game G_A, played by two players I and II. First I chooses a natural number a_0, then II chooses a natural number b_0, then I chooses a_1, then II chooses b_1, and so on. The game ends after ω steps; if the resulting sequence $\langle a_0, b_0, a_1, b_1, \ldots \rangle$ is in A, then I wins, otherwise II wins. A strategy (for I or II) is a rule that tells the player what move to make depending on the previous moves of both players. A strategy is a winning strategy if the player who follows it always wins. The game G_A is determined if one of the players has a winning strategy.

The Axiom of Determinacy (AD) states that for every $A \subset \omega^\omega$, the game G_A is determined.

[5] Jech, Set Theory: The Third Millennium Edition.

Some consequences of AD

- every set of reals is Lebesgue Measurable;
- every set reals has the Perfect Set Property;
- the club filter on ω_1 is an ultrafilter;

The last fact obviously implies that GL is not the logic of the club topology, but gives a hint that the logic of the club filter could be GL.3 (see the blackboard).

Club topology

We have:

Theorem (Aguilera, S.)

If AD *holds, then the logic of the club topology is* GL.3*.*

The proof employs a useful fact about GL.3, namely:

Fact

If $\mathsf{L} \supset \mathsf{GL.3}$ *is non-degenerate, i.e. is consistent with any* *worm*[a] *of the form* $\Diamond\top, \Diamond\Diamond\top, \ldots, \Diamond\ldots\Diamond\top$, *then* $\mathsf{L} = \mathsf{GL.3}$*;*

[a] A variable-free (poly-)modal formula.

This boils down any proof of completeness to showing soundness and showing consistency with the worms.

Normal topology

Another interesting topological semantics introduced by Blass is based on normal complete ultrafilters:

Definition (Measure)

Let $F \subset P(\kappa)$ be a filter, we say that F is a (normal) measure if it is κ-closed, i.e. for any collection $\{A_i : i < \gamma\} \subset F$ for $\gamma < \kappa$, the intersection $\bigcap A_i \in F$ and normal, i.e. for any $\{A_i : i < \kappa\} \subset F$, the set:

$$\Delta_i A_i = \{\alpha : \alpha \in \bigcup_{i<\alpha} A_i\} \in F$$

The normal topology of κ is defined using the filter F_N^κ, where

$$A \in F_n^\kappa \text{ iff } \forall F \subset P(\kappa)(F \text{ is a normal measure} \to A \in F)$$

Completeness for the normal topology

Theorem (Golshani, Zoghifard)

It is consistent[a] *that* GL *is the logic of the normal topology.*

[a] From infinitely many strong cardinals.

Theorem (Aguilera, S.)

If $V = L[\mathcal{U}]$*, where* $\mathcal{U}$ *is a sequence of measures, such that there is a measure of Mitchell rank* n *for all* $n < \omega$*, then the logic of the normal topology is* GL.3*. In fact*[a]*, the logic of any* mouse*, that has measures of all finite Mitchell ranks, is* GL.3*.*

[a] Follows from a result of Farmer Schlutzenberg.

Mice finally meet worms

The fact that there are arbitrary large finite Mitchell ranks is needed to show that the logic is consistent with any worm $\Diamond^n\top$. Thus:

If a mouse has worms of arbitrary length, then the logic of the normal topology is GL.3.

Bouquet-topology

The ω-bouquet approach gives other nice generalizations:

Theorem (Golshani, S., Zoghifard)

The logic GL *with λ-many variables is complete with respect to λ-bouquets and with respect to the subspaces of the order topology.*

Theorem (S.)

GL.3 *is strogly complete with respect to ultralinear bouquets.*

See the blackboard.

Completeness results for GLP.3

The canonical topologies

The axioms of GLP(.3) are defined on polytopological spaces $(X, \tau_i)_{i<\omega}$. With the properties that for any $i < j < \omega$:

- τ_i is scattered;
- $\tau_i \subset \tau_j$;
- for any $A \subset X$, $d_{\tau_i} A \in \tau_j$;

Thus we have a natural way to generate topologies starting with τ_0 being the order topology. For each τ we define $\tau^+ = \tau \cup \{d_\tau A : A \subset X\}$ and let $\tau_{i+1} = \tau_i^+$.

The canonical topologies

The canonical way is to take the ordinals with their order topology as the initial space. The issue is that starting with τ_2, it is consistent with ZFC that the topology is discrete. In general for the topology τ_n to be non-discrete we need existence of *n-s-reflecting* cardinal.[6].

	name	θ_n	$d_n(A)$
τ_0	order	ω	$\{\alpha \in \mathrm{Lim} : A \cap \alpha \text{ is unbounded in } \alpha\}$
τ_1	club	ω_1	$\{\alpha : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \alpha\}$
τ_2	Mahlo	θ_2	$\{\alpha : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is } 2 - \text{stationary in } \alpha\}$
$\vdots$	$\vdots$	$\vdots$	......
τ_k	...	θ_k	$\{\alpha : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is } k - \text{stationary in } \alpha\}$
$\vdots$	$\vdots$	$\vdots$	......

[6] Beklemishev and Gabelaia, "Topological interpretations of provability logic".

Axiom of determinacy

Theorem (Aguilera, Jackson(?), S.)

If $V = L(\mathbb{R}) + \mathrm{AD}$*, then the logic of* $(\mathrm{Ord}, \tau_{2i})_{i<\omega}$ *is* GLP.3 *and the logic of* $(\mathrm{Ord}, \tau_{2i+1})_{i<\omega}$ *is* GLP.

Thank you all!