FINE STRUCTURE AND MICE WITH WOODIN CARDINALS IN A NUTSHELL. THE MOUSE M_ω

G. STEPANOV

Contents

1. Extenders

Definition 1. Let $\kappa < \lambda$ and suppose that M is transitive and rudimentarily closed. We call E a (κ, λ)-extender over M iff there is a nontrivial Σ_0-elementary embedding $j : M \to N$, with N transitive and rudimentarily closed, such that $\kappa = \text{crit}(j), \lambda < j(\kappa)$, and

$$E = \left\{(a, x) : a \in [\lambda]^{<\omega} \wedge x \subseteq [\kappa]^{|a|} \wedge x \in M \wedge a \in j(x)\right\}.$$

We say in this case that E is derived from j, and write $\kappa = \text{crit}(E), \lambda = \text{lh}(E)$.

We have compatibility, normality and each E_a is a normal κ-complete ultrafilter.

Given an extender E then $\text{Ult}(M, E)$ is constructed by setting

$$\langle a, f\rangle \sim \langle b, g\rangle \iff \{u \in [\kappa]^{|a\cup b|} : f_{a,a\cup b}(u) = g_{b,a\cup b}(u)\} \in E_{a\cup b}$$

then the element of the model are corresponding classes of equivalence $[a, f]^M_E$ with

$$[a, f]^M_E \in [b, g]^M_E \iff \{u \in [\kappa]^{|a\cup b|} : f_{a,a\cup b}(u) \in g_{b,a\cup b}(u)\} \in E_{a\cup b}$$

If $M \models \mathsf{AC}$ then Łoś's theorem hold for Σ_0 formulæ, that is, given φ a Σ_0 formula

$$\text{Ult}(M, E) \models \varphi([a_1, f_1], \ldots, [a_n, f_n])$$

iff.

$$\{u \in [\kappa]^c : M \models \varphi((f_1)_{a_1,c}(u), \ldots, (f_n)_{a_n,c}(u))\} \in E_c$$

where $c = \bigcup a_i$. Hence, the canonical embedding possesses Σ_1-elementarity, where the canonical embedding $i^M_E : M \to \text{Ult}(M, E) : x \mapsto [\{0\}, c_x]$ with $c_x : [\kappa]^1 \to M : u \mapsto x$.

Given a (κ, λ)-pre-extender over M and $\xi \leq \lambda$ we set $E|\xi = \{(a, x) \in E : a \subset \xi\}$. Then we can define a natural embedding $\sigma([a, f]^M_{E|\xi}) = [a, f]^M_E$. We call ξ a *generator* of E if $\xi = \text{CRIT}(\sigma)$, that is $\xi \neq [a, f]^M_E$ for all $f \in M$ and $a \subset \xi$.

Date: October 24, 2023 (compiled).

Definition 2. Given E is a (κ, λ)-pre-extender over M, then

$$\nu(E) = \sup(\kappa^{+M} \cup \{\xi + 1 : \xi \text{ is a generator of } E\})$$

we call $\nu(E)$ the *support* of E.

2. Potential pre-mouse and their fine structure

Now we specify some desired properties of extenders to procreate mice.
We work with the J hierarchy
TO DEFINE
with $J_\alpha^{\vec{E}} = J_\alpha^A$, where $A = \{(\beta, z) : z \in E_\beta\}$.

Definition 3. A set A is acceptable at α iff.

$$\forall \beta < \alpha \forall \kappa ((P(\kappa) \cap (J_{\beta+1}^A \setminus J_\beta^A) \neq \emptyset) \to J_{\beta+1}^A \models |J_\beta^A| \leq \kappa)$$

If A is acceptable in α and $J_\alpha^A \models$»κ^+ exists«, thenx $J_\alpha^A \models$»$P(\kappa)$ exists and $P(\kappa) \subset J_{\kappa^+}^A$«, hence GCH holds there.

Let E be a pre-extender over M, and $M \models$»κ^+ exists«, where $\kappa = \text{CRIT}(E)$. Let $\nu = \nu(E)$ and $\eta = (\nu^+)^{\text{Ult}(M,E)}$ is in the wfp of $\text{Ult}(M, E)$. Now let E^* be the (κ, η)-pre-extender of derived from E. Then $\nu = \nu(E^*)$, so that $E|\nu = E^*|\nu$ and the pre-extenders are equivalent. We call E^* the trivial completion of E. And we index E as η.

We shall use another technical concept. Let E be an extender over M. We say that it is of *type* Z if $\nu(E) = \lambda + 1$ for some limit λ such that $\lambda = \nu(E|\lambda)$ and $(\lambda^+)^{\text{Ult}(M,E)} = (\lambda^+)^{\text{Ult}(M,E|\lambda)}$. In this case E^* as well as $(E|\lambda)^*$ shall be indexed as at the same place. Hence we do not allow E to be of type Z.

Definition 4. A *fine extender sequence* is a sequence $\vec{E}$ such that for each $\alpha \in \text{dom}\vec{E}$, $\vec{E}$ is acceptable in α and either $E_\alpha = \emptyset$ or E_α is (κ, α)-pre-extender over $J_\alpha^{\vec{E}}$ for some κ such that $J_\alpha^{\vec{E}} \models$»$\kappa^+$ exists«, amd:

1. E_α is the trivial completion of $E_\alpha|\nu(E_\alpha)$, and hence $\alpha = (\nu(E_\alpha)^+)^{\text{Ult}(J_\alpha^{\vec{E}}, E_\alpha)}$ and E_α is not of a type Z.
2. (Coherence) $i(\vec{E}|\kappa)|\alpha = \vec{E}|\kappa$ and $i(\vec{E}|\kappa)_\alpha = \emptyset$, where $i : J_\alpha^{\vec{E}} \to \text{Ult}(J_\alpha^{\vec{E}}, E_\alpha)$ is the canonical embedding, and
3. (Closure under initial segment) for any η such that $(\kappa^+)^{J_\alpha^{\vec{E}}} \leq \eta < \nu(E_\alpha)$, $\eta = \nu(E_\alpha|\eta)$, and $E_\alpha|\nu$ is not of type Z, one of the things holds:
   (a) there is a $\gamma < \alpha$ such that E_γ is the trivial completion of $E_\alpha|\eta$, or
   (b) $E_\eta \neq \emptyset$ and letting $j : J_\eta^{\vec{E}} \to \text{Ult}(J_\eta^{\vec{E}}, E_\eta)$ be canonical embedding and $\mu = \text{CRIT}(j)$, there is a $\gamma < \alpha$ such that $j(\vec{E}|\mu)_\gamma$ is the trivial completion of $E_\alpha|\nu$.

Definition 5. A *potential premouse* (or ppm) is a structure of the form $(J_\alpha^{\vec{E}}, \in, \vec{E}|\alpha, E_\alpha)$, where $\vec{E}$ is a fine extender sequence. We use $\mathcal{J}_\alpha^{\vec{E}}$ to denote this structure.

Definition 6. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ be a ppm. We say that $\mathcal{M}$ is *active* if $E_\alpha \neq \emptyset$, and *passive* otherwise. If it is passive, then we say $\nu = \nu(E_\alpha)$ and $\kappa = \text{CRIT}(E_\alpha)$, we say it is *type* I if $\nu = (\kappa^+)^{\mathcal{M}}$, is *type* II if ν is a successor ordinal and is *type* III if ν is a limit ordinal $> (\kappa^+)^{\mathcal{M}}$.

Definition 7. A structure $(M, \in, A_1, A_2, \dots)$ is *amenable* if

$$\forall X \in M \forall i (A_i \cap x \in M)$$

We want amenability to be satisfied. We can do it by encoding E_α as E^c_α is a set of quadruples (γ, ξ, a, x) such that

$$(\nu(E_\alpha) < \gamma < \alpha) \wedge (\text{CRIT}(E_\alpha) < \xi < \text{CRIT}(E_\alpha)^+)^{J^{\vec{E}}_\alpha})$$
$$\wedge (E_\alpha \cap ([\nu(E_\alpha)]^{<\omega} \times J^{\vec{E}}_\xi)) \wedge ((a, x) \in (E_\alpha \cap ([\gamma]^{<\omega} \times J^{<\omega}_\xi)))$$

This makes the structure $(J^{\vec{E}}_\alpha, \in, \vec{E}|\alpha, E^c_\alpha)$ amenable.

Codes and projecta.

Definition 8. $\mathcal{L}$ is the language of set theory with additional constant symbols $\dot{\mu}, \dot{\nu}, \dot{\gamma}$, and additional unary predicate symbols $\dot{E}$ and $\dot{F}$.

If $\mathcal{M}$ is active, then we set

$$\mu^{\mathcal{M}} = \text{CRIT}(E_\alpha)$$

If $\mathcal{M}$ is active of a type II, then there is the longest non-type-Z initial segment F of E_α containing properly less information than E_α itself, and we let $\gamma^{\mathcal{M}}$ determine where F appears on $\vec{E}$ or an ultrapower of $\vec{E}$. More precisely we set

$$F = \begin{cases} (E_\alpha|(\nu^{\mathcal{M}} - 1))^* & \text{, if } ()^* \text{is not type Z} \\ (E_\alpha|\nu(E_\alpha|(\nu^{\mathcal{M}} - 1)) - 1)^* & \text{, otherwise} \end{cases}$$

$$\gamma = \text{the unique } \xi \text{ with } E_\xi = F$$

if no such then $\eta = \nu(F)$, which we have by the definition of 3b

Definition 9. Let $\mathcal{M} = \mathcal{J}^{\vec{E}}_\alpha$ be a ppm; then the Σ_0 *code* $\mathcal{M}$ or $\mathcal{C}_0(\mathcal{M})$ is the $\mathcal{L}$-structure given by:

1. (1) if $\mathcal{M}$ is passive then $\mathcal{N}$ has universe $J^{\vec{E}}_\alpha$, $\dot{E}^{\mathcal{N}} = \vec{E}|\alpha$, $\dot{F}^{\mathcal{N}} = \emptyset$, and $\dot{\mu}^{\mathcal{N}} = \dot{\nu}^{\mathcal{N}} = \dot{\gamma}^{\mathcal{N}} = 0$;
2. (2) if $\mathcal{M}$ is active of types I or II, then $\mathcal{N}$ has universe $J^{\vec{E}}_\alpha, \dot{E}^{\mathcal{N}} = \vec{E}\lceil\alpha$, $\dot{F}^{\mathcal{N}} = E^*_\alpha$ (where E^*_α is the amenable coding of E_α), and $\dot{\mu}^{\mathcal{N}} = \mu^{\mathcal{M}}$, $\dot{\nu}^{\mathcal{N}} = \nu^{\mathcal{M}}$, and $\dot{\gamma}^{\mathcal{N}} = \gamma^{\mathcal{M}}$;
3. (3) if $\mathcal{M}$ is active type III, then letting $\nu = \nu(E_\alpha), \mathcal{N}$ has universe $J^{\vec{E}}_\nu$, $\dot{E}^{\mathcal{N}} = \vec{E}|\nu, \dot{F}^{\mathcal{N}} = E_\alpha|\nu, \dot{\mu}^{\mathcal{N}} = \mu^{\mathcal{M}}$, and $\dot{\nu}^{\mathcal{N}} = \dot{\gamma}^{\mathcal{N}} = 0$;

Definition 10. Let $\mathcal{M}$ be a ppm; then we call the least ordinal α such that there is some $\mathbf{\Sigma}^{\mathcal{C}_0(\mathcal{M})}_1$ subset $A \subset \alpha$ with $A \notin \mathcal{C}_0(\mathcal{M})$, the Σ_1 projectum of $\mathcal{M}$ or $\rho_1(\mathcal{M})$. (In particular $\rho_1(\mathcal{M}) \leq \mathsf{Ord} \cap \mathcal{C}_0(\mathcal{M})$.)

Notice that the new set A may not be (lightface) Σ_1-definable. Since there is a $\Sigma^{\mathcal{C}_0(\mathcal{M})}_1$ map from the class of finite sets of ordinals onto $\mathcal{C}_0(\mathcal{M})$, we can take the parameter from which A is defined to be a finite set of ordinals. We standardize the parameter by minimizing it in a certain wellorder.

Definition 11. A parameter is a finite sequence $\langle \alpha_0, \dots, \alpha_n \rangle$ of ordinals such that $\alpha_0 > \cdots > \alpha_n$ (and could be empty). If $\mathcal{M}$ is a ppm, then the first standard parameter of $\mathcal{M}$, or $p_1(\mathcal{M})$, is the lexicographically least parameter p such that there is a $\Sigma^{\mathcal{C}_0(\mathcal{M})}_1(\{p\})$ set A such that $(A \cap \rho_1(\mathcal{M})) \notin \mathcal{C}_0(\mathcal{M})$

Definition 12. (1) For any $\mathcal{L}$ structure $\mathcal{Q}$ and set $X \subset |\mathcal{Q}|$, $\mathcal{H}_1^{\mathcal{Q}}(X)$ is the transitive collapse of the substructure of $\mathcal{Q}$ whose consists of all $y \in |\mathcal{Q}|$, such that $\{y\}$ is $\Sigma_1^{\mathcal{Q}}$ definable from parameters in X.

(2) for any ppm $\mathcal{M}$, the *first core* of $\mathcal{M}$, $\mathcal{C}_1(\mathcal{M})$, is defined by: $\mathcal{C}_1(\mathcal{M}) = \mathcal{H}_1^{\mathcal{C}_0(\mathcal{M})}(\rho_1(\mathcal{M}) \cup \{p_1(\mathcal{M})\})$.

For $\mathcal{C}_1(\mathcal{M})$ exists $\mathcal{N}$, such that $\mathcal{C}_0(\mathcal{N}) = \mathcal{C}_1(\mathcal{M})$. It follows from $\mathcal{C}_1(\mathcal{M}) \prec_{\Sigma_1} \mathcal{C}_0(\mathcal{M})$ and saying »I am a code« is Π_2, hence downwards absolute.

Definition 13. Let $\mathcal{M}$ be a ppm.

(1) We say that $p_1(\mathcal{M})$ is universal if whenever $A \subset \rho_1(\mathcal{M})$ and $A \in C_0(\mathcal{M})$, then $A \in C_1(\mathcal{M})$;
(2) Let $p_1(\mathcal{M}) = \langle \alpha_0, \dots, \alpha_n \rangle$. We say that $p_1(\mathcal{M})$ is 1-*solid* if whenever $i \leq n$ and A is $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}(\{\alpha_0, \dots, \alpha_{i-1}\})$, then $A \cap \alpha_i \in \mathcal{C}_0(\mathcal{M})$;
(3) We say that $\mathcal{M}$ 1-*solid* just in case $p_1(\mathcal{M})$ is 1-solid and 1-universal;

Lemma 14. *If $\mathcal{N}$ is such that $\mathcal{C}_1(\mathcal{M}) = \mathcal{C}_0(\mathcal{N})$ and $p_1(\mathcal{M})$ is universal, then $\rho_1(\mathcal{M}) = \rho_1(\mathcal{N})$ and the image of $p_1(\mathcal{M})$ under transitive collapse is $p_1(\mathcal{N})$.*

Proof. Let r be the image of $p_1(\mathcal{M})$ under transitive collapse. First, if $\alpha < \rho_1(\mathcal{M})$ then $\alpha < \rho_1(\mathcal{N})$ and hence $\rho_1(\mathcal{M}) \leq \rho_1(\mathcal{N})$. Now consider $s <_{\text{lex}} r$, if A is definable over $\mathcal{C}_0(\mathcal{N})$, by minimality of $p_1(\mathcal{M})$ is in $\mathcal{C}_0(\mathcal{M})$, hence in $\mathcal{C}_0(\mathcal{N})$.

Moreover, as the collapse is identity on $\rho_1(\mathcal{M})$, r defines a new Σ_1 subset of $\rho_1(\mathcal{M})$ over $\mathcal{C}_0(\mathcal{N})$, hence $\rho_1(\mathcal{N}) \leq \rho_1(\mathcal{M})$ and $p_1(\mathcal{N}) \leq_{\text{lex}} r$. □

The 1-solidity of $p_1(\mathcal{M})$ is important in showing that $i\,(p_1(\mathcal{M})) = p_1(\mathcal{Q})$ for certain ultrapower embeddings $i : \mathcal{M} \to \mathcal{Q}$.

Lemma 15. *Let $\mathcal{C}_0(\mathcal{M})$ be a ppm with $\rho_1(\mathcal{M}) < \mathsf{Ord}^{\mathcal{M}}$. If r is 1-solid, then $r \leq_{\text{lex}} p = p_1(\mathcal{M})$.*

Proof. Let $p = \langle \alpha_0, \dots, \alpha_n \rangle$ and $r = \langle \beta_0, \dots, \beta_n \rangle$. Suppose $p < r$. Let d be the first disagreement of p and r, i.e. $\alpha_i = \beta_i$ for $i < d$ and $\alpha_d < \beta_d$. Or $d = n + 1$, then r is an end extension. Using this fact any A which is $\Sigma_1^{\mathcal{C}_0(\mathcal{M})}(\{\beta_0, \dots, \beta_{d-1}\})$ does not define any new set A with $\beta_d \cap A \notin \mathcal{C}_0(\mathcal{M})$. □

Inductively we can define n-projecta, soundness, solidity.
So,

Definition 16. Let $\mathcal{M}$ be a ppm; then $\mathcal{M}$ is ω-solid iff. $\mathcal{M}$ is n-solid for all $n < \omega$, and $\mathcal{M}$ is ω-sound iff. $\mathcal{M}$ is n-sound for all $n < \omega$. If $\mathcal{M}$ is ω-solid, then we let $\rho_\omega(\mathcal{M})$ be the eventual value of $\rho_n(\mathcal{M})$ and $\mathcal{C}_\omega(\mathcal{M})$ the eventual value of $\mathcal{C}_n(\mathcal{M})$ as $n \to \omega$.

Definition 17. Let $\mathcal{M} = \mathcal{J}_\alpha^{\vec{E}}$ be a ppm, let $\beta \leq \alpha$; we write $\mathcal{J}_\beta^{\mathcal{M}}$ for $\mathcal{J}_\beta^{\vec{E}}$. Then $\mathcal{N}$ is an initial segment iff. $\exists \beta \leq \alpha (\mathcal{N} = \mathcal{J}_\beta^{\mathcal{M}})$.

We call ppm a *coded premouse* if all its initial segments are ω-sound.

Definition 18. Let E be a (κ, λ)-extender over $\mathcal{C}_0(\mathcal{M})$; then we say that E is *close to* $\mathcal{C}_0(\mathcal{M})$ iff. for every $a \in [\lambda]^{<\omega}$

(1) E_a is Σ_1-definable over $\mathcal{C}_0(\mathcal{M})$ form parameter, and
(2) if $\in \mathcal{C}_0(\mathcal{M})$ and $\mathcal{C}_0(\mathcal{M}) \models |\mathcal{A}| \leq \kappa$, then $E_a \cap \mathcal{A} \in \mathcal{C}_0(\mathcal{M})$;

Fine structure and Ultrapowers. So far the construction of ultrapower could only guarantee us Σ_1-elementarity. If $\mathcal{M}$ is active, n-sound and $\text{CRIT}(E) < \rho_n(\mathcal{M})$, then we can generate a stronger ultrapower of $\mathcal{M}$, one for which Łoś's theorem holds for $r\Sigma_n$ formulæ. Roughly speaking, one allowed to use $r\Sigma_n$-definable functions with parameters from $\mathcal{C}_0(\mathcal{M})$. Since $\text{CRIT}(E) < \rho_n(\mathcal{M})$, we can say that E measures enough sets for such an embedding.

Definition 19. Let $\pi : \mathcal{C}_0(\mathcal{M}) \to \mathcal{C}_0(\mathcal{N})$ and let $n \leq \omega$. We call π an *n-embedding* if

(1) $\mathcal{M}$ and $\mathcal{N}$ are n-sound;
(2) π is $r\Sigma_n$-elementary;
(3) $\pi(p_i(\mathcal{M})) = p_i(\mathcal{N})$ for all $i \leq n$;
(4) $\pi(\rho_i(\mathcal{M})) = \rho_i(\mathcal{N})$ for all $i < n$ and $\sup(\pi[\rho_n(\mathcal{M})]) = \rho_n(\mathcal{N})$;

We call π an ω-embedding if it is fully elementary, it preserves projecta and parameters.

Lemma 20. *For all $n \leq \omega$ canonical embedding associated with n-th ultrapower is an n-embedding.*

Corollary 21. *Let $\mathcal{M}$ be a premouse. And let E be a (κ, λ)-extender over $\mathcal{C}_0(\mathcal{M})$, which is close to $\mathcal{C}_0(\mathcal{M})$ with $\kappa < \rho_n(\mathcal{M})$. Let $\mathcal{N}$ be such that $\mathcal{C}_0(\mathcal{N}) = \text{Ult}_n(\mathcal{C}_0(\mathcal{M}), E)$, $\mathcal{M}$ is n-sound and $(n+1)$-solid, and $\rho_{n+1}(\mathcal{M}) \leq \kappa$, then $\mathcal{N}$ is n-sound but not $(n+1)$-sound.*

3. Iteration trees

Now given a k-sound premouse and θ is an ordinal, we define the *iteration game* $\mathcal{G}_k(\mathcal{M}, \theta)$.

Definition 22. A *tree order on α* is a strict partial order T of α with least element 0 such that for all $\gamma < \alpha$:

(1) $\beta T \gamma \implies \beta < \gamma$
(2) $\{\beta : \beta T \gamma\}$ is wellordered by T.
(3) γ is a successor ordinal $\iff$ γ is a T-successor, and
(4) γ is a limit ordinal $\implies$ $\{\beta : \beta T \gamma\}$ is cofinal in γ.

If T is a tree order then

$$[\beta, \gamma]_T = \{\eta : \eta = \beta \vee \beta T \eta T \gamma \vee \eta = \gamma\}$$

we say that premice $\mathcal{M}$ and $\mathcal{N}$ *agree below* γ iff. $\mathcal{J}_\beta^{\mathcal{M}} = \mathcal{J}_\beta^{\mathcal{M}}$ for all $\beta < \gamma$.

The Game. Players are given a tree T of order θ, a premice M_α for $\alpha < \theta$ with $\mathcal{M}_0 = \mathcal{M}$, an extender F_α form the $\mathcal{M}_\alpha$ sequence, and a set $D \subset \theta$ and embedding $i_{\alpha,\beta} : \mathcal{C}_0(\mathcal{M}_\alpha) \to \mathcal{C}_0(\mathcal{M}_\beta)$ defined whenever $\alpha T \beta$ and $D \cap (\alpha; \beta]_T = \emptyset$.

The rules guarantee that $\alpha \leq \beta \implies \mathcal{M}_\alpha$ agrees with $\mathcal{M}_\beta$ below $\text{lh}(F_\alpha)$ and $\alpha < \beta \implies \text{lh}(F_\alpha)$ is a cardinal of $\mathcal{M}_\beta$ (see the proof later).

At move $\alpha + 1$ the player I picks an extender F_α from the $\mathcal{M}_\alpha$ with $\text{lh}(F_\xi) < \text{lh}(F_\alpha)$ for all $\xi < \alpha$ (if they cannot, the game is over and they loses). Now let $\beta \leq \alpha$ be the least such that $\text{CRIT}(F_\alpha) < \nu(F_\beta)$. Let now

$$\mathcal{M}^*_{\alpha+1} = \mathcal{J}_\gamma^{\mathcal{M}_\beta}, \text{ where } \gamma \text{ is the largest } \eta, \text{ such that } F_\alpha \text{ is a pre-extender over } \mathcal{J}_\eta^{\mathcal{M}_\beta}.$$

Our agreement hypothesis imply that γ exists, $\mathrm{lh}(F_\beta) \leq \gamma$, and F_α is a pre-extender over $\mathcal{C}_0(\mathcal{J}_\gamma^{\mathcal{M}_\beta})$.

Proof. It is clear if $\beta = \alpha$. Let now $\beta < \alpha$ and $\kappa = \textsc{crit}(F_\alpha)$. Having $\mathrm{lh}(F_\beta) < \mathrm{lh}(F_\alpha)$ and $\mathrm{lh}(F_\beta)$ being a cardinal if $\mathcal{M}_\alpha$.

$$P(\kappa) \cap \mathcal{J}^{\mathcal{M}_\beta}_{\mathrm{lh}(F_\beta)} = P(\kappa) \cap \mathcal{M}_\alpha = P(\kappa) \cap \mathcal{J}^{\mathcal{M}_\alpha}_{\mathrm{lh}(F_\alpha)}$$

□

Then we put $\alpha + 1 \in D$ iff. $\mathcal{M}^*_{\alpha+1}$ is a proper initial segment of $\mathcal{M}_\beta$.

Len now $n \leq \omega$ be the largest that: 1) $\textsc{crit} F_\alpha < \rho_n(M^*_{\alpha+1})$ and 2) if $D \cap [0; \alpha + 1]_T = \emptyset$, then $n \leq k$, we set

$$\mathcal{M}_{\alpha+1} = \mathrm{Ult}(\mathcal{M}^*_{\alpha+1}, F_\alpha)$$

if it is well-defined, otherwise II has lost. Finally we let $\beta T(\alpha+1)$, and if $\alpha+1 \notin D$, then $i_{\beta,\alpha+1} : \mathcal{C}_0(\mathcal{M}_\beta) \to \mathcal{C}_0(\mathcal{M}_{\alpha+1})$ is the canonical ultrapower embedding.

It can be shown that the hypothesis is retained.

Player II acts on limits by picking a cofinal (in λ) well-founded branch, that it $D \cap b$ is bounded in λ and the limit (in end-segment) is well-founded. Then we set $\mathcal{M}_\gamma$ and embedding $i_{\alpha,\lambda}$ for all $\alpha \in b \setminus \sup b$.

If they reach θ, player II wins.

Definition 23. A *k-maximal iteration tree on* $\mathcal{M}$ is a partial play of the game $\mathcal{G}_k(\mathcal{M}, \theta)$ in which neither player has yet lost.

Lemma 24. *Let $\mathcal{T}$ be an iteration tree, and let $\alpha + 1 < \mathrm{lh}(\mathcal{T})$; then E_α is close to $\mathcal{M}^*_{\alpha+1}$.*

Definition 25. If $\mathcal{T}$ is an iteration tree with models $\mathcal{M}_\alpha$ and extenders E_α, and $\alpha + 1 < \mathrm{lh}(\mathcal{T})$, then $\deg^{\mathcal{T}}(\alpha + 1)$ is the largest $n \leq \omega$ such that $\mathcal{M}_{\alpha+1} = \mathrm{Ult}_n\left(\mathcal{M}^*_{\alpha+1}, E_\alpha\right)$. Also, we use $i^{*\mathcal{T}}_{\alpha+1}$ for the canonical embedding from $\mathcal{M}^*_{\alpha+1}$ into this ultrapower.

Definition 26. A (k, θ)-iteration strategy for $\mathcal{M}$ is a winning strategy for II in $\mathcal{G}_k(\mathcal{M}, \theta)$. We say $\mathcal{M}$ is (k, θ)-iterable iff. there is such a strategy.

Comparison.

Definition 27. 3.10 Definition. A branch b of the iteration tree $\mathcal{T}$ drops (in model or degree) iff $D^{\mathcal{T}} \cap b \neq \emptyset$ or $\deg^{\mathcal{T}}(b) < \deg^{\mathcal{T}}(0)$.

Theorem 28 (The Comparison Lemma)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be k-sound premice of size $\leq \theta$, and suppose that Σ and Γ are $(k, \theta^+ + 1)$-iteration strategies for $\mathcal{M}$ and $\mathcal{N}$ respectively; then there are iteration trees $\mathcal{T}$ and $\mathcal{U}$ played according to Σ and Γ; and having last models $\mathcal{M}^{\mathcal{T}}_\alpha$ and $\mathcal{N}^{\mathcal{U}}_\eta$ such that either*

(1) $[0, \alpha]_T$ does not drop in model or degree, and $\mathcal{M}^{\mathcal{T}}_\alpha$ is an initial segment of $\mathcal{N}^{\mathcal{U}}_\eta$;
(2) or vice versa;

Corollary 29. *Let $\mathcal{M}$ and $\mathcal{N}$ be ω-sound $(\omega, \omega_1 + 1)$-iterable premice such that $\rho_\omega(\mathcal{M}) = \rho_\omega(\mathcal{N}) = \omega$; then $\mathcal{M}$ is an initial segment of $\mathcal{N}$, or vice-versa.*

Proof. By comparison Lemma we have trees $\mathcal{T},\mathcal{U}$ and w.l.o.g. $\mathcal{M}_\alpha^\mathcal{T} \lhd \mathcal{N}_\beta^\mathcal{U}$. Then for no extender $\text{CRIT}(E) < \rho_\omega(\mathcal{M})$, hence $[0,\alpha]$ does not drop iff. $\alpha = 0$. If $\beta > 0$ then $\mathcal{N}_\beta^\mathcal{U}$ is not ω-sound. Thus, $\mathcal{M}$ is a proper initial segment, which is countable in $\mathcal{N}_\beta^\mathcal{U}$, hence $\mathcal{M}$ is a proper initial segment of $\mathcal{N}$, since iteration does not produce new reals. □

Corollary 30. *If $\mathcal{M}$ and $\mathcal{N}$ are (ω,ω_1+1)-iterable premice, then the $\mathcal{M}$ constructibility order on $\mathbb{R}\cap\mathcal{M}$ is an initial segment of the $\mathcal{N}$-constructibility order on $\mathbb{R}\cap\mathcal{N}$, or vice-versa.*

Proof. Let $\langle x_i : i < \gamma\rangle = \mathbb{R}\cap\mathcal{M}$ and $\langle y_i : i < \beta\rangle$ let $\gamma \leq \delta$. Let j be the least ordinal such that $x_j \neq y_j$. Let $x_j \in \mathcal{J}_{\alpha+1}^\mathcal{M} \setminus \mathcal{J}_\alpha^\mathcal{M}$ and $x_j \in \mathcal{J}_{\beta+1}^\mathcal{M} \setminus \mathcal{J}_\beta^\mathcal{N}$, then $\rho_\omega(\mathcal{J}_\alpha^\mathcal{M}) = \rho_\omega(\mathcal{J}_\beta^\mathcal{N}) = \omega$, now we apply the previous lemma, having w.l.o.g. $\mathcal{J}_\alpha^\mathcal{M} = \mathcal{J}_{\beta'}^\mathcal{N}$ for $\beta' \leq \beta$. Then x_j appears as some $F_i(a_1,\dots,a_n)$, but then this x_j should also appear in $\mathcal{J}_{\beta'+1}^\mathcal{N}$, since $E\cap J_\alpha^\mathcal{M} = F\cap J_{\beta'}^\mathcal{N}$. □

4. Condensation and solidity

Theorem 31. *Let $\mathcal{M}$ be ω-sound and $(\omega,\omega_1,\omega_1+1)$-iterable. Suppose that $\pi : \mathcal{H}\to\mathcal{M}$ is fully elementary, and $\text{crit}(\pi) = \rho_\omega^\mathcal{H}$; then either*

1. *$\mathcal{H}$ is a proper initial segment of $\mathcal{M}$, or*
2. *there is an extender E on the $\mathcal{M}$-sequence such that $\text{lh}(E) = \rho_\omega^\mathcal{H}$, and $\mathcal{H}$ is a proper initial segment of $\text{Ult}_0(\mathcal{M},E)$.*

Theorem 32. *Let $k < \omega$, and let $\mathcal{M}$ be a k-sound, (k,ω_1,ω_1+1)-iterable premouse; then $\mathcal{C}_{k+1}(\mathcal{M})$ exists, and agrees with $\mathcal{M}$ below γ, for all γ of $\mathcal{M}$-cardinality $\rho_{k+1}(\mathcal{M})$.*

Theorem 33. *Let $\mathcal{M}$ be an $(\omega,\omega_1,\omega_1+1)$-iterable premouse satisfying the axioms of ZF, except perhaps Power Set; then the following are true in $\mathcal{M}$:*

1. *for all uncountable regular $\kappa, \diamondsuit_\kappa$;*
2. *for all uncountable regular κ ($\diamondsuit_\kappa^+ \iff \kappa$ is not ineffable);*
3. *for all infinite cardinals $\kappa, \square_\kappa$;*

5. Mice with Woodin cardinals

Definition 34. A premouse $\mathcal{M}$ is ω-*small* if whenever κ is the critical point of an extender of $\mathcal{M}$-seuqnece, then

$$\mathcal{J}_\kappa^\mathcal{M} \not\models \text{»there are } \omega \text{ Woodin cardinals«}$$

Definition 35. $M_\omega^\#$ is the unique $(\omega,\omega_1,\omega_1+1)$-iterable mouse which is not ω-small, but all its initial segments are.

We can see that, it projects to ω, hence countable and uniqueness holds.