Strong Kripke Completeness of the Closed Fragment of GLP

AiML 2024

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August 2024

Epigraph

'The very idea of "logic" disintegrates in the vortex of a more original questioning. . . ' Martin Heidegger

Provability logic

The logic GLP

Logic GLP is the smallest set of formulæ in $\mathcal{L}_\Box$ closed under modus ponens, that contains classical tautologies and modal axioms which reflect provability nature of the Boxes:

1. $[n](p \to q) \to ([n]p \to [n]q)$ (Normality)
2. $[n]([n]p \to p) \to [n]p$ (Löb)
3. $[m]p \to [n][m]p, m \leq n$
4. $\langle m \rangle p \to [n]\langle m \rangle p, m < n$
5. $[m]p \to [n]p, m \leq n$

It is Kripke incomplete! (See the blackboard if you can).

Arithmetical completeness

Fix some gödelian theory T and some valuation function

$$v : \mathrm{var} \to \mathcal{L}_{PA}$$

this can yield an arithmetical interpretation:

$$[\![p]\!]_T = v(p) \quad [\![\varphi \wedge \psi]\!]_T = [\![\varphi]\!]_T \wedge [\![\psi]\!]_T$$

$$[\![\neg\varphi]\!]_T = \neg[\![\varphi]\!]_T \quad [\![[n]\varphi]\!]_T = \mathsf{Pr}_{T+\Sigma^0_n}(\overline{\ulcorner [\![\varphi]\!]_T \urcorner})$$

where $\overline{n}$ is a numeral, that it $\overline{n} = s^n(0)$.

$$Log(T) = \{\varphi \in \mathcal{L} : \forall(v : \mathrm{var} \to \mathcal{L}_{PA})\, T \vdash [\![\varphi]\!]_T\}$$

It was shown that $Log(\mathrm{PA}) = \mathrm{GLP}$ as well as some other natural interpretations (due to Japaridze).

General problems

1. it is ‘the’ provability logic;
2. it is quite capricious;
3. no Kripke completeness;
4. topological completeness is extremely tricky;

That is why people study its fragments, which are sometimes interesting enough not only to be helpful for modal logic related problems, but also with proof theory, ordinal analysis and so on.

Ignatiev logic

Definition

Logic I is the smallest set of formulæ in $\mathcal{L}_\Box$ closed under modus ponens, that contains classical tautologies and the modal axioms:

1. $[n](p \to q) \to ([n]p \to [n]q)$ (Normality)
2. $[n]([n]p \to p) \to [n]p$ (Löb)
3. $[m]p \to [n][m]p, m \leq n$
4. $\langle m \rangle p \to [n]\langle m \rangle p, m < n$
5. ~~$[m]p \to [n]p, m \leq n$~~

There are non-trivial Kripke models for this logic.

Some facts

Let $\mathcal{L}_0$ be the variable free (poly)modal language. Then

Fact (Ignatiev)

$\mathsf{GLP} \cap \mathcal{L}_0 = \mathsf{I} \cap \mathcal{L}_0$.

Fact (Ignatiev)

The closed fragmanet of GLP *is Kripke complete, moreover it is Kripke complete w.r.t. a single frame, namely the Ignatiev frame $\mathfrak{I}$ (see next slides). Indeed, it is Kripke complete w.r.t to a designated set of points, namely the main axis.*

I-frames

We say that a Kripke frame is an I-*frame*, if each R_i is converse well-founded and transitive and the following holds:

$$\forall x, y\big(xR_ny \to \forall z(xR_mz \leftrightarrow yR_mz)\big)$$

if $m < n$.

Ordinals

Definition

ε_0 is the least ordinal α that satisfies the equation $\omega^\alpha = \alpha$.

The ordinal ε_0 satisfies

$$\varepsilon_0 = \sup \Big\{ \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}}_{n \text{ times}} : n < \omega \Big\} = \omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{\cdot^{\cdot^{\cdot}}}}}}}}.$$

Definition

Given an ordinal $\alpha > 0$, it can be uniquely represented in its *Cantor normal form* as

$$\alpha = \omega^{\lambda_0} + \omega^{\lambda_1} + \cdots + \omega^{\lambda_n}$$

where $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$. Assuming $\alpha > 0$ we define $\log(\alpha) = \lambda_n$ and its iteration, $\log^0(\alpha) = \alpha$ and $\log^{n+1}(\alpha) = \log(\log^n(\alpha))$. Similarly, we put $\log(0) = 0$.

The $\mathfrak{I}$-frame

Definition

Let $\iota \leq \varepsilon_0$. The Ignatiev frame $\mathfrak{I}_{\leq\iota}$ consists of functions

$$\vec{\alpha} : \omega \to \iota + 1$$

with the property that $\alpha_{i+1} \leq \log(\alpha_i)$. Here and below, we write α_i to mean $\alpha(i)$ in order to regard $\vec{\alpha}$ as a sequence. For $\vec{\alpha}, \vec{\beta} \in \mathfrak{I}_{\leq\iota}$, we define $\vec{\alpha} R_k \vec{\beta}$ if and only if the following hold:

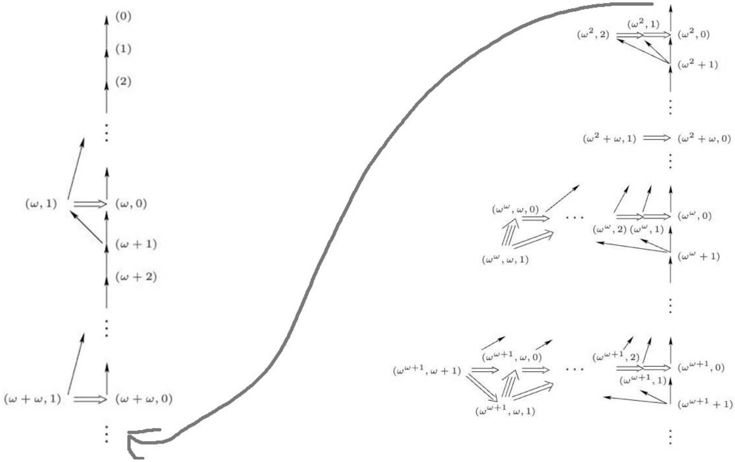
- $\forall i < k \ \alpha_i = \beta_i$;
- $\alpha_k > \beta_k$.

Definition

Let ι be an ordinal. We define $\mathfrak{I}_{<\iota}$ to be the union of $\{\mathfrak{I}_{\leq\eta} : \eta < \iota\}$.

Behold!

The picture has been smuggled (and jacked up) from PhD thesis of Joosten:



Completeness results

Now we can state the result by Ignatiev more precisely:

Fact (Ignatiev)

Given a closed formula φ*, we have* $\mathsf{GLP} \vdash \varphi \leftrightarrow \mathfrak{I}_{<\varepsilon_0} \models \varphi$.

Theorem

Given a set of closed formulæ Γ *consistent with* GLP*, there is a point* $\vec{\alpha} \in \mathfrak{I}_{\leq \varepsilon_0}$*, such that* $\mathfrak{I}, \alpha \Vdash \Gamma$.

Even more precisely

Theorem (Strong completeness)

(A) *The closed fragment of* GLP *is strongly complete with respect to* $\mathfrak{I}_{\leq\varepsilon_0}$. *More precisely: let* Γ *be a set of closed* $\mathcal{L}$*-formulæ. Then the following are equivalent:*

- (i) Γ *is consistent with* GLP*; and*
- (ii) $\mathfrak{I}_{\leq\varepsilon_0}, \vec{\alpha} \Vdash \Gamma$ *for some* $\vec{\alpha} \in \mathfrak{I}_{\leq\varepsilon_0}$.

(B) *Moreover,*

- (i) *The closed fragment of* GLP *is not strongly complete with respect to* $\mathfrak{I}_{<\varepsilon_0}$*; and*
- (ii) *The closed fragment of* GLP *is not strongly complete with respect to the main axis of* $\mathfrak{I}_{\leq\varepsilon_0}$.

Key lemmas and ideas

The proof sketch

Fix maximal GLP-consistent $\Gamma \subset \mathcal{L}$.

For each $i < \omega$, we let

$$\Gamma_i = \Big\{[i]\varphi : [i]\varphi \in \Gamma\Big\} \cup \Big\{\langle i\rangle\psi : \langle i\rangle\psi \in \Gamma\Big\}.$$

The plan is to treat these strata consecutively and show inductively that for each $k < \omega$ there exists α_k such that:

1. if $k \neq 0$, then $\alpha_i \leq \log(\alpha_{i-1})$ for each $0 < i < k$, and
2. $\mathfrak{I}_{\leq \epsilon_0}, \vec{\gamma} \Vdash \bigcup_{i \leq k} \Gamma_i$ whenever $\gamma_i = \alpha_i$ for each $i \leq k$.

If we have it, then $\langle \alpha_i \rangle_{i<\omega} \Vdash \Gamma$.

Lemma 3.5

The following lemma shows uniqueness of the choices of α's.

Lemma

For each $k < \omega$ and $\alpha < \epsilon_0$ there exists a closed formula (indeed worm) φ_k^α, such that whenever $\vec{\beta} \in \mathfrak{I}_{\leq \epsilon_0}$ we have $\vec{\beta} \Vdash \langle k \rangle \varphi_k^\alpha$ if and only if $\beta_k > \alpha$. Moreover, for each $\alpha \leq \epsilon_0$ there exists a set of formulæ T_k^α such that for each $\vec{\gamma}$, $\vec{\gamma} \Vdash T_k^\alpha$ if and only if $\gamma_k = \alpha$.

- $\alpha = 0$ then $\varphi_0^\alpha = \top$;
- $\alpha = \beta + 1 + \gamma$ for some $\beta \geq \gamma$, then $\varphi_0^\alpha = \varphi_0^\gamma \langle 0 \rangle \varphi_0^\beta$;
- $\alpha = \omega^\beta$ then $\varphi_0^\alpha = \uparrow\varphi_0^\beta$;[1]

For $\alpha < \varepsilon_0$ we have $T_0^\alpha = \{\langle 0 \rangle \varphi^{\alpha'} : \alpha' < \alpha\} \cup \{\neg \langle 0 \rangle \varphi^\alpha\}$ and $T_0^{\epsilon_0} = \{\langle 0 \rangle \varphi^{\alpha'} : \alpha' < \varepsilon_0\}$.

[1] A similar construction was used by Beklemishev, Fernandez-Duque, Joosten.

Bad news (or is it?)

Counterexamples

Lemma

Let $\Gamma = \{\langle 0\rangle^k \top : k < \omega\} \cup \{[1]\bot\} \cup \{[0][1]\bot\}$. *Then,* Γ *is consistent with* GLP, *but for all* $\vec{\alpha} \in \mathrm{ma}(\mathfrak{I}_{\leq\epsilon_0})$, *we have* $\mathfrak{I}_{<\epsilon_0}, \vec{\alpha} \nVdash \Gamma$. *Moreover* Γ *cannot be satisfied at any point of any Icard or Beklemishev-Gabelaia spaces*[a]*.*

[a] Nonetheless, some recent developments by Aguilera and S. yield some hope to circumvent this predicament.

Proof.

Of the first statement It is easy to verify directly that $\mathfrak{I}_{\leq\epsilon_0}, \langle\omega, 0\rangle \Vdash \Gamma$, so that indeed Γ is consistent with GLP. Suppose $\mathfrak{I}_{<\epsilon_0}, \vec{\alpha} \Vdash \Gamma$. Then by $\langle 0\rangle^k \top$, we must have $\alpha_0 > k$. By $[1]\bot$, we must have $\alpha_1 = 0$ and by $[0][1]\bot$ we must have $\alpha_0 \leq \omega$, so the only point in $\mathfrak{I}_{\leq\epsilon_0}$ which satisfies Γ is $\langle\omega, 0\rangle$, which is not on the main axis of $\mathfrak{I}_{\leq\epsilon_0}$. □

Thank you all!