Stationary reflection under AD and provability logic

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Provability logic

The logic GL

Logic GL is the smallest set of formulæ in $\mathcal{L}_\Box$ closed under modus ponens, that contains classical tautologies and modal axioms which reflect provability nature of the Box:

$$\Box(p \to q) \to (\Box p \to \Box q) \text{ (Normality)}$$

$$\Box(\Box p \to p) \to \Box p \text{ (Löb)}$$

It is known to be sound and complete w.r.t. the class of finite irreflexive trees.

Arithmetical completeness

Fix some gödelian theory T and some valuation function

$$v : \mathrm{var} \to \mathcal{L}_{PA}$$

this yields an arithmetical interpretation:

$$[\![p]\!]_T = v(p) \quad [\![\varphi \wedge \psi]\!]_T = [\![\varphi]\!]_T \wedge [\![\psi]\!]_T$$

$$[\![\neg\varphi]\!]_T = \neg[\![\varphi]\!]_T \quad [\![\Box\varphi]\!]_T = \mathsf{Pr}_T(\overline{\ulcorner [\![\varphi]\!]_T \urcorner})$$

where $\overline{n}$ is a numeral, that it $\overline{n} = s^n(0)$.

$$Log(T) = \{\varphi \in \mathcal{L}_\Box : \forall (v : \mathrm{var} \to \mathcal{L}_{PA}) T \vdash [\![\varphi]\!]_T\}$$

A celebrated result by Solovay[1] shows that $Log(\mathrm{PA}) = \mathrm{GL}$.

[1] Solovay, "Provability interpretations of modal logic".

The logic GLP

Logic GLP is the smallest set of formulæ in $\mathcal{L}_{[0],[1],\ldots}$ closed under modus ponens, that contains classical tautologies and modal axioms which reflect provability nature of the Boxes:

1. $[n](p \to q) \to ([n]p \to [n]q)$ (Normality)
2. $[n]([n]p \to p) \to [n]p$ (Löb)
3. $[m]p \to [n][m]p, m \leq n$
4. $\langle m \rangle p \to [n]\langle m \rangle p, m < n$
5. $[m]p \to [n]p, m \leq n$

This logic is also arithmetically complete, yet it is Kripke incomplete.

Topological semantics

Definitions

Definition (Derivative operator)

Given a topological space (X, τ) for each $A \subset X$ we denote $d_\tau A = \{x : \forall U \in \tau \exists y \neq x (y \in U \cap A)\}$.

Definition

A *topological model* is a pair (X, τ, v), where (X, τ) is a topological space and $v : \mathrm{Vars} \to \mathcal{P}(X)$ is an interpretation of variables. Such an interpretation is extended to arbitrarily formulæ as follows:

- $[\![p]\!] = v(p)$; $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$; $[\![\Diamond\varphi]\!] = d_\tau[\![\varphi]\!]$;

Interval topology

One can show that GL is valid in a space (X, τ) if and only if it is *scattered*, i.e., if every $A \subset X$ has an isolated point. A natural example of such space is an ordinal with its order topology (also called interval topology). In fact, GL is complete for such semantics and indeed it suffices to restrict to a single ordinal:

Fact (Abashidze, Blass)

[a][b] GL *is sound and complete with respect to every ordinal* $\Omega \geq \omega^\omega$ *when equipped with the order topology.*

[a] Abashidze, "Ordinal completeness of the Gödel-Löb modal system".
[b] Blass, "Infinitary combinatorics and modal logic".

GLP-topologies

A natural polymodal generalization of the logic GL is the logic GLP, which infinitely many Box-operators corresponding to different strictly increasing notions of provability. In particular $\mathsf{GLP} \vdash [n]\varphi \to [n+1]\varphi$ and $\mathsf{GLP} \vdash \langle n \rangle \varphi \to [n+1]\langle n \rangle \varphi$, this forces a polytopological space $(X, \tau_n)_{n<\omega}$ to have the folloing properties:

- τ_n is scattered for each $n < \omega$;
- $d_{\tau_n} A \in \tau_{n+1}$ for each $A \subset X$ and $n < \omega$;
- $\tau_n \subset \tau_{n+1}$ for each $n < \omega$;

The most natural approach to build such a space is to take some scattered (X, τ) and apply the following operation iteratively:

$$\tau \mapsto \tau^+ := \tau \cup \{d_\tau A : A \subset X\}$$

The canonical topologies

The canonical way is to take the ordinals with their order topology as the initial space. The issue is that starting with τ_1, it is consistent with ZFC that the topology is discrete. In general for the topology τ_n to be non-discrete we need existence of *n-s-reflecting* cardinal.[2].

	name	θ_n	$d_n(A)$
τ_0	order	ω	$\{\alpha \in \mathsf{Lim} : A \cap \alpha \text{ is unbounded in } \alpha\}$
τ_1	club	ω_1	$\{\alpha : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is stationary in } \alpha\}$
τ_2	Mahlo	θ_2	$\{\alpha : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is } 2 - \text{stationary in } \alpha\}$
$\vdots$	$\vdots$	$\vdots$	$\ldots\ldots$
τ_k	$\ldots$	θ_k	$\{\alpha : \mathrm{cf}(\alpha) > \omega \text{ and } A \cap \alpha \text{ is } k - \text{stationary in } \alpha\}$
$\vdots$	$\vdots$	$\vdots$	$\ldots\ldots$

[2] Beklemishev and Gabelaia, "Topological interpretations of provability logic".

Generalised stationarity

n-s-stationary sets

Definition

Let κ be a cardinal of uncountable cofinality, we say that $S \subset \kappa$ is 0-*stationary* if it is unbounded in κ, say that it is ξ-*stationary* if for any $\zeta < \xi$ and any ζ-stationary T, there is $\alpha \in S$, such that $S \cap \alpha$ is ζ-stationary in α.

One can see that $S \subset \kappa$ is 1-stationary if and only if it is stationary.

Definition

Let κ be a cardinal of uncountable cofinality, we say that $T \subset \kappa$ is 0-*s-stationary reflecting*, if it is unbounded in κ, say that it is ξ-*s-stationary reflecting*, if for any $\zeta < \xi$ and any two ζ-stationary sets S and R, there is $\alpha \in T$, such that $S \cap R \cap \alpha$ is ζ-stationary in α.

AD consequences

n-s-stationary reflecting cardinals are large cardinals for $n > 1$, thus completeness of GLP or its fragments is independent of ZFC. We focus on the AD-setting. The following basic fact

Fact (AD)

The club filter on ω_1 is an ultrafilter.

shows that GL is not complete for (Ord, τ_1). However,

Theorem (Aguilera, S., (AD))

$\mathrm{Log}((\aleph_{\varepsilon_0}, \tau_1)) = \mathrm{GL.3}$.

Where

$$\mathrm{GL.3} = \mathrm{GL} + \Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \vee \psi \to \varphi)$$

where the latter is the axiom of linearity.

Thus, the further goal is to understand the generalized stationary reflection under AD.

Fact (Essentially Steve Jackson)

Let κ be a regular cardinal below $\aleph_{\omega_1}$, then for any regular $\lambda < \kappa$, the λ-club filter in an ultrafilter (in fact a measure).

Obviously, the club filter is an intersection of λ-club filters for all regular $\lambda < \kappa$. The fact thus implies that a set is stationary if it is a member of a λ-club filter for some regular $\lambda < \kappa$. This allows a convenient characteriation of 2-stationary sets, one can easily see that $S \subset \kappa$ is 2-stationary if the set $\{\lambda : S \text{ is a } \lambda\text{-club}\}$ is unbounded in the regular cardinals below κ. It follows that the first stationary reflection cardinal is the ω-th regular cardinal, which (under AD) is $\boldsymbol{\delta}^1_{\omega+1} = \aleph_{\varepsilon_0+1}$.

Result and perspectives

Results

Results:

1. $\mathrm{Log}((\aleph_{\varepsilon_0}, \tau_2)) = \mathrm{GL.3}$;
2. $\rho_\omega = \boldsymbol{\delta}^1_{\omega+1} = \aleph_{\varepsilon_0+1}$ is the first stationary reflecting cardinal, where ρ_α is the α's regular cardinal;

Conjectures:

1. the first $(i+2)$-stationary reflecting cardinal is ρ_{κ_i}, where κ_i is the first i-reflecting cardinal;
2. $\mathrm{Log}((\mathrm{Ord}, \tau_0, \tau_2, \ldots, \tau_{2n}, \ldots)) = \mathrm{GLP}$;
3. $\mathrm{Log}((\mathrm{Ord}, \tau_1, \tau_3, \ldots, \tau_{2n+1}, \ldots)) = \mathrm{GLP.3}$;

Thank you all!

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