On topological properties of quantum channels

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Introduction

Introduction

Motivations:

- Stinespring and Kraus representation are crucial in studying quantum channels
- proper topology should be natural in physical sense
- Stinespring and Kraus representations are multy-valued
- continuous selection is a natural and desired property

Preliminaries

C^*-algebras

Definition

C^*-algebra is a Banach algebra with an involution operation $\cdot^*$, which possesses the following properties:

- $(x+y)^* = x^* + y^*, \quad (xy)^* = x^* y^*$
- $(\lambda x)^* = \bar{\lambda} x^*$
- $\|x^* x\| = \|x\| \|x^*\|$

Self-adjoint ($x = x^*$) elements of a C^*-algebra are called *positive* and the set of all positive elements we denote by V^+. We say that $a \geq b$ iff. $a - b \in V^+$.

Completely-positive maps

Definition

A linear map A between C^*-algebras is called *positive* if it maps positive elements to positive, A is a C^**-homomorphism* if it is positive and preserves squares of self-adjoint elements.

Definition

A linear map A between C^*-algebras is called completely positive, the map $\mathrm{id}_n \otimes A$ is positive for any $n \in \mathbb{N}$, where $\mathrm{id}_n : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is an identity.

Quantum channels

Definition

Given Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ and algebras of operators $\mathfrak{B}(\mathcal{H}_A), \mathfrak{B}(\mathcal{H}_B)$, *quantum channel* in the Schrödinger picture is a map $\Phi : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_B)$ that is trace preserving and completely positive (cptp). Dually, quantum channel in the Heisenberg picture is a completely positive unital map (cpu) $\Phi^* : \mathfrak{B}(\mathcal{H}_B) \to \mathfrak{B}(\mathcal{H}_A)$.

There is a dual cpu Φ^* for any cptp Φ and vice versa, defined by $\mathrm{Tr}\rho\Phi^*(C) = \mathrm{Tr}\Phi(\rho)A$ for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$, $C \in \mathfrak{B}(\mathcal{H}_B)$.

Representations

The next two theorems provides useful characterisation of quantum channels:

Theorem

(Stinespring) For any cptp map Φ *there exists a Hilbert space* $\mathcal{H}_E$ *and isometry* $V : \mathcal{H}_A \to \mathcal{H}_{BE}$*, such that* $\Phi(\rho) = Tr_E V \rho V^*$ *and* $\Phi^*(C) = V^*(C \otimes I_E)V$ *for any* $\rho \in \mathfrak{S}(\mathcal{H}_A)$*,* $C \in \mathfrak{B}(\mathcal{H}_B)$*.*

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Theorem

(Kraus) For any cptp map Φ *there exists a set of operators* $V_i : \mathcal{H}_A \to \mathcal{H}_B$*, such that* $\Phi(\rho) = \sum_i V_i \rho V_i^*$ *and* $\Phi^*(C) = \sum_i V_i^* C V_i$ *for any* $\rho \in \mathfrak{S}(\mathcal{H}_A)$*,* $C \in \mathfrak{B}(\mathcal{H}_B)$*, moreover* $\sum_i V_i^* V_i = I$*.*

Topologies on quantum channels

There are several natural topologies on the set of quantum channels:

Definition

Diamond norm is the following functional
$\|\Phi\|_\diamond = \sup_{\rho \in \mathfrak{S}(\mathcal{H}_{AR})} \|\Phi \otimes \mathrm{Id}_R(\rho)\|_1$

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Consider the single-mode Bosonic quantum limited attenuator, uniquely defined by $\mathcal{A}_\eta(|\alpha\rangle\langle\alpha|) = |\eta\alpha\rangle\langle\eta\alpha|$. Consider one-parametric family of attenuators $\{\mathcal{A}_{e^{-t}}\}_{t \in \mathbb{R}}$, by [1] $\|\mathcal{A}_\eta - \mathcal{A}_{\eta'}\|_\diamond = 2$, whenever $\eta \neq \eta'$ and $|\eta| \leq 1$.

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Definition

Strong topology is a topology, defined by the family of seminorms $\Phi \mapsto \|\Phi(\rho)\|_1$.

Results

Preliminaries

Definition

Choi rank of a quantum channel is the minimal cardinality of the set of operators in Kraus decomposition. (Or minimal dimension of $\mathcal{H}_E$ in Stinespring representation).

Theorem

(Choi–Jamiołkowski isomorphism) Let $\mathcal{H}, \mathcal{H}'$ and $\mathcal{K}$ be separable Hilbert spaces and $|\Omega\rangle$ be an unit vector in $\mathcal{H} \otimes \mathcal{K}$ such that $\sigma = \mathrm{Tr}_{\mathcal{H}} |\Omega\rangle\langle\Omega|$ is a full rank state in $\mathcal{K}$. Then

$$\mathfrak{H} : \mathfrak{F}_{\leq 1}(\mathcal{H}, \mathcal{H}') \to \mathfrak{T}(\mathcal{H}') \otimes \mathfrak{T}(\sigma) \ : \ \Phi \mapsto \Phi \otimes Id(|\Omega\rangle\langle\Omega|)$$

is a topological isomorphism. (With respect to strong topology on the set on cp maps). [2]

Preliminaries

For given full rank state $\sigma = \sum_i \lambda_i |i\rangle\langle i|$ in $\mathfrak{S}(\mathcal{K})$ let $\mathfrak{T}(\sigma)$ be the subset of $\mathfrak{T}_1(\mathcal{K})$ consisting of all operators A such that $\sum_{i,j} \frac{\langle i|A|j\rangle}{\sqrt{\lambda_i \lambda_j}} |i\rangle\langle j| \leq I_{\mathcal{K}}$.

By $\mathfrak{T}(\mathcal{H}) \otimes \mathfrak{T}(\sigma)$ we assume $\{A \in \mathfrak{T}_1(\mathcal{H} \otimes \mathcal{K}) \mid \operatorname{Tr}_{\mathcal{H}} A \in \mathfrak{T}(\sigma)\}$.

Result

Proposition

Given $n \in \mathbb{N}$, then the set of channels with Choi rank not exceeding n is closed in strong topology.

(Proof sketch) By Choi–Jamiołkowski isomorphism we only need to show closedness of $\mathfrak{H}$-image of this set.

$$\Phi_k \to_s \Phi \iff \Phi_k \otimes Id(|\Omega\rangle\langle\Omega|) \to_{\|\cdot\|_1} \Phi \otimes Id(|\Omega\rangle\langle\Omega|)$$

$$\iff \lim_k \sum_{i \leq n} V_k^{(i)} \otimes I|\Omega\rangle\langle\Omega|V_k^{(i)*} \otimes I = A$$

By the argument from proof in [2], if A has rank n, then its $\mathfrak{H}$-preimage has Choi rank at most n.

Result

The following lemma completes the proof:

Lemma

Let $\langle \sigma_k : k \in \mathbb{N} \rangle \subset \mathfrak{S}(\mathcal{H})$ be such that $\sigma_k \to_{\|\cdot\|_1} \sigma$ and σ_k has rank at most N, than σ also has rank at most N.

Result

Definition

Channel Φ is *unitarily equivalent to a partial trace*, if there exists $\mathcal{H}_E$ and unitary $V : \mathcal{H}_A \to \mathcal{H}_{BE}$, such that $\Phi^*(C) = V^*(C \otimes I_E)V$.

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Proposition

Quantum channel is unitarily equivalent to a partial iff.
$\Phi^*(C^*C) = \Phi^*(C^*)\Phi^*(C)$ *for any* $C \in \mathfrak{B}(\mathcal{H})$.

Preliminaries

KSW theorem provides continuity of Stinespring representation in the following form:

$$\frac{1}{2}\|\Phi - \Psi\|_\diamond \leq \inf \|V_\Phi - V_\Psi\| \leq \sqrt{\|\Phi - \Psi\|_\diamond}$$

In [3] selective continuity for sequences was shown. But general selective continuity is unclear for Stinespring/unitary representations. Here we provide finite-dimensional version of this property.

Result

Proposition

Let $F \subset \mathfrak{F}(\mathcal{H}_A; \mathcal{H}_B)$ be a finite-dimensional simplex, defined by a convex hull of the set of points $\{\Phi_i : i < n\}$. Let $G \subseteq F$ be a (diamond norm) connected subset of F. Then there exists a continuous selection for Stinespring dilation for this set.

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Consider $\mathrm{rest} : V \mapsto V^*((\cdot) \otimes I_E)V$. The proof idea is based on the fact that any point $\varphi \in G$ can be uniquely represented as a convex combination $\sum_{i<n} \lambda_i \Phi_i$. Then we consider arbitrary representation $\{V_i : i < n\}$, such that $\mathrm{rest}(V_i) = \Phi_i$. In this case we can take $\bigoplus_{i<n} \sqrt{\lambda_i} V_i$ as a preimage of such φ.

Further steps

Conjecture

Continuous selection exists for any compact set of quantum channels.

Steps might be taken

- generalize lemma for countable simplices
- deal with uniqueness of representation
- apply analog of Gelfand-Mazur theorem

Summary

Summary

- closedness of the set of channels with bounded Choi-rank was shown
- criterion for channels unitarily equivalent to a partial trace was shown
- restricted property of continuous selection for Stinespring dilation was shown

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