Completeness of GL.3 for normal topology

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Provability logic

The logic GL

Logic GL.3 is the smallest set of formulæ in $\mathcal{L}_\Box$ closed under modus ponens, that contains classical tautologies and modal axioms which reflect provability nature of the Box:

$$\Box(p \to q) \to (\Box p \to \Box q) \text{ (Normality)}$$

$$\Box(\Box p \to p) \to \Box p \text{ (Löb)}$$

It is known to be sound and complete w.r.t. the class of finite irreflexive trees.

Arithmetical completeness

Fix some gödelian theory T and some valuation function

$$v : \mathrm{var} \to \mathcal{L}_{PA}$$

this yields an arithmetical interpretation:

$$[\![p]\!]_T = v(p) \qquad [\![\varphi \wedge \psi]\!]_T = [\![\varphi]\!]_T \wedge [\![\psi]\!]_T$$

$$[\![\neg\varphi]\!]_T = \neg[\![\varphi]\!]_T \quad [\![\Box\varphi]\!]_T = \mathsf{Pr}_T(\overline{\ulcorner [\![\varphi]\!]_T \urcorner})$$

where $\overline{n}$ is a numeral, that it $\overline{n} = s^n(0)$.

$$Log(T) = \{\varphi \in \mathcal{L}_\Box : \forall (v : \mathrm{var} \to \mathcal{L}_{PA}) T \vdash [\![\varphi]\!]_T\}$$

A celebrated result by Solovay[1] shows that $Log(\mathrm{PA}) = \mathrm{GL}$.

[1] Solovay, "Provability interpretations of modal logic".

Model-(set)-theoretic completeness for GL.3

Fix d some valuation function

$$v : \mathrm{var} \to \mathcal{L}_{\mathsf{ZFC}}$$

this yields the following interpretation:

$$[\![p]\!] = v(p) \quad [\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$$

$$[\![\neg\varphi]\!] = \neg[\![\varphi]\!] \quad [\![\Box\varphi]\!] = \text{“}[\![\varphi]\!] \text{ holds at all } V_\kappa \text{ where } V_\kappa \models \mathsf{ZFC}\text{”}$$

$$Log(\mathsf{ZFC}) = \{\varphi \in \mathcal{L}_\Box : \forall(v : \mathrm{var} \to \mathcal{L}_{\mathsf{ZFC}})(V \models [\![\varphi]\!])\}$$

In the same article[2] Solovay showed that this logic is GL.3.

[2] Solovay, “Provability interpretations of modal logic”.

The logic GL.3

$$\mathsf{GL.3} = \mathsf{GL} + \Box(\Box p \to q) \vee \Box(\Box q \wedge q \to p)$$

It is known to be sound and complete with respect to the class of finite linear frames.

Proof of soundness.

On the blackboard. □

Topological semantics

Definitions

Definition (Derivative operator)

Given a topological space (X, τ) for each $A \subset X$ we denote $d_\tau A = \{x : \forall U \in \tau \exists y \neq x (y \in U \cap A)\}$.

Definition

A *topological model* is a pair (X, τ, v), where (X, τ) is a topological space and $v : \mathrm{Vars} \to \mathcal{P}(X)$ is an interpretation of variables. Such an interpretation is extended to arbitrarily formulæ by the following clauses:

- $[\![p]\!] = v(p)$; $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$; $[\![\Diamond\varphi]\!] = d_\tau[\![\varphi]\!]$;

Interval topology

One can show that GL is valid in a space (X, τ) if and only if it is *scattered*, i.e., if every $A \subset X$ has an isolated point. A natural example of such space is an ordinal with its order topology (also called interval topology). In fact, GL is complete for such semantics and indeed it suffices to restrict to a single ordinal:

Fact (Abashidze, Blass)

[a][b] GL *is sound and complete with respect to every ordinal* $\Omega \geq \omega^\omega$ *when equipped with the order topology.*

[a] Abashidze, "Ordinal completeness of the Gödel-Löb modal system".
[b] Blass, "Infinitary combinatorics and modal logic".

Some facts and conventions (cw: abuse of notation)

Definition

Given a cardinal κ, we say that $U \subset P(\kappa)$ is a normal measure if it is a non-principal κ-complete normal ultrafilter.

Fact

Given an uncountable ordinal α, the set of closed and unbounded in α subsets of α is a normal filter.

Warning: throughout this talk "is measure one" or "in a(n ultra)filter" or "is an open set" will sometimes be used synonymously.

Ordinal spaces

Every extension of order topology is of course a model of GL. There are some natural examples of them:

- closed unbounded (club) topology[3] – τ_C
  (i.e. $A \subset \kappa$ is a punctured neigborhood of κ iff A extends some closed unbounded subset of κ);
- normal topology[4] – τ_U
  (i.e. $A \subset \kappa$ is a punctured neigborhood of κ iff $A \in \bigcap\{U : U \text{ is a normal measure on } \kappa\}$;
- subtle, ineffable, indescribable, etc.

Expectedly completeness is independent for most of the cases.

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[3] It *is* consistent that GL is complete to it, Blass, "Infinitary Combinatorics and Modal Logic"

[4] It *is* consistent that GL is complete to it, Golshani and Zoghifard, "Completeness of the Gödel–Löb Provability Logic for the Filter Sequence of Normal Measures"

Why study them?

Reason 1:
GLP!

Reason 2:

Fun fact

Let κ be measurable with $\mu(A) = 1$ for some normal measure μ and some $A \subset \kappa$, then $\mu'(\{\alpha \in \kappa : \alpha \cap A \text{ is measure 0 for all measures on } \alpha\}) = 1$ for some normal measure μ' on κ.

Fun fact

Let $S \subset \kappa$ (regular uncountable) be a stationary set, then $S' = \{\alpha : \alpha \cap S \text{ is not stationary in } S\}$ is stationary as well.

Both follow from the second incompleteness theorem.

Proof

Indeed, we have $\mathrm{Con}(T + \varphi)$ implies $\mathrm{Con}(T + \varphi + \neg\,\mathrm{Con}(T + \varphi))$, by completeness it follows that $\mathrm{GL} \vdash \Diamond p \to \Diamond(p \wedge \Diamond\neg p)$.
Assume GL is sound for (X, τ), then for any $x \in X$ we have $X, x \Vdash \Diamond p \to \Diamond(p \wedge \Diamond\neg p)$. Now take

$$\tau = \tau_C \text{ or}$$
$$\tau = \tau_U.$$

Q.E.D.

Completeness result

Soundness!

The key lemma is the following:

Lemma

Suppose ZFC *holds and* κ *is an ordinal such that the Mitchell order on normal measures on cardinals* $<\kappa$ *is linear. Let* v *be a topological interpretation on* (κ, τ_U)*. For each* α, β*, let* $U_{\alpha,\beta}$ *denote the unique normal measure on* α *of order* β*, if it exists. Then, the following are equivalent:*

1. $\xi \in v(\Box\varphi)$ *for* $U_{\alpha,\beta}$*-a.e.* $\xi < \alpha$*, and*
2. $\xi \in v(\varphi)$ *for* $U_{\alpha,\gamma}$*-a.e.* $\xi < \alpha$*, for each* $\gamma < \beta$*.*

Soundness!!

Lemma

Assume $V = L[\mathcal{U}]^a$*, where* $\mathcal{U}$ *is a set of measures. Then* GL.3 *is validated by* (Ord, τ_U)*.*

[a] The universe is the constructible universe built relative to $\mathcal{U}$.

Proof.

A similar case distinction. See the blackboard. □

Completeness

The key fact that makes completeness almost trivial is the following:

Fact

Assume L *is a non-degenerate extension of* GL.3 *(i.e.* $\mathsf{L} \supset \mathsf{GL.3}$ *and* $\Box^n\bot \notin \mathsf{L}$ *for any* n*), then* $\mathsf{L} = \mathsf{GL.3}$*.*

Taking $\mathcal{U}$ that has measures with Mitchell order n for each n we get

Theorem

Assume $V = L[\mathcal{U}]$*, where* $\mathcal{U}$ *is such that for each* n *there is a measure of Mitchell order* n*. Then* $\mathrm{Log}(\mathrm{Ord}, \tau_U) = \mathsf{GL.3}$*.*

Moreover

Theorem

Assume the Axiom of Determinacy, then $\mathrm{Log}(\aleph_{\varepsilon_0}, \tau_C) = \mathrm{GL.3}$.

[5]

[5] Follows from certain structural consequences of AD and so on.

Thank you all!

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