Non-determenistic semantics for modal logic

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Introduction

Introduction

Motivations:

- philosophical: do possible worlds really exist?[1]
- finite models: reasoning about decidability/complexity
- possible application of SAT-solvers

[1] John T. Kearns. "Modal Semantics without Possible Worlds". In: *J. Symb. Log.* 46.1 (1981), pp. 77–86.

Preliminaries

Modal language

Definition

Consider a countable set of propositional variables $Prop = \{p_i \mid i < \omega\}$. Then $\mathcal{L}$ is defined as follows:

$$\mathcal{L} = p_i \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \Box\varphi$$

where $p_i \in Prop, \varphi, \psi \in \mathcal{L}$.

Nmatrices

A Nmatrix M for $\mathcal{L}$ is a triple of the form $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a set of *truth values*
- $\mathcal{D} \subseteq \mathcal{V}$ is a set of *designated truth values*
- $\mathcal{O}$ is a function assigning a *truth table* $\mathcal{V}^n \to P(\mathcal{V}) \setminus \{\emptyset\}$ to every n-ary connective $\diamond$ of $\mathcal{L}$

In the context of Nmatrix $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we often denote $\mathcal{O}(\diamond)$ by $\tilde{\diamond}$.[2]

[2] Lahav and Zohar, "Effective Semantics for the Modal Logics K and KT via Non-deterministic Matrices".

Example of Nmatrix for $\mathbb{K}$

Definition

Nmatrix $\mathrm{M}_{\mathbb{K}} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, with $\mathcal{V} = \{\mathrm{t}, \mathrm{f}, \mathrm{F}, \mathrm{T}\}$ $\mathcal{D} = \{\mathrm{t}, \mathrm{T}\}$ and where $\mathcal{O}$ is defined by the following tables:

$x \tilde{\wedge} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
t	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
F	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
f	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$

$x \tilde{\vee} y$	T	t	F	f
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
F	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$
f	$\mathcal{D}$	$\mathcal{D}$	$\overline{\mathcal{D}}$	$\overline{\mathcal{D}}$

x	$\tilde{\neg} x$
T	$\overline{\mathcal{D}}$
t	$\overline{\mathcal{D}}$
F	$\mathcal{D}$
f	$\mathcal{D}$

x	$\tilde{\Box} x$
T	$\mathcal{D}$
t	$\overline{\mathcal{D}}$
F	$\mathcal{D}$
f	$\overline{\mathcal{D}}$

The intuition behind these truth-values is the following:

- $v(\varphi) = \mathrm{f}$ if φ doesn't hold in w and doesn't hold in some possible world;
- $v(\varphi) = \mathrm{t}$ if φ holds in w but doesn't hold in some possible world;
- $v(\varphi) = \mathrm{F}$ if φ doesn't hold in w, but holds in all possible worlds;
- $v(\varphi) = \mathrm{T}$ if φ holds in w and holds in all possible worlds;

Legal valuations

Given $\mathcal{F} \subseteq \mathcal{L}$, then an $\mathcal{F}$-valuation $v : \mathcal{F} \to \mathcal{V}$ is *M-legal* if $v(\varphi) \in \text{pos-val}(\varphi, M, v)$ for every formula $\varphi \in \mathcal{F}$ whose immediate subformulas are contained in $\mathcal{F}$, where $\text{pos-val}(\varphi, M, v)$ is defined by:

1. $\text{pos-val}(p, M, v) = \mathcal{V}$ for every atomic formula p.
2. $\text{pos-val}(\diamond(\psi_1, \ldots, \psi_n), M, v) = \tilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$ for every non-atomic formula $\diamond(\psi_1, \ldots, \psi_n)$.

Definitions of $\models$ and $\vdash$

Definition

Let $\mathcal{V}$ be the set of truth-values and its proper subset $\mathcal{D}$ – the set of *designated values*. Consider a (possibly non-total) valuation $v : \mathcal{L} \to \mathcal{V}$ and a formula $\varphi \in \mathrm{Dom}(v) \subseteq \mathcal{L}$, then we write $v \models_{\mathcal{D}} \varphi$, if $v(\varphi) \in \mathcal{D}$. For $\Sigma \subseteq \mathrm{Dom}(v)$ we write $v \models_{\mathcal{D}} \Sigma$ if $v \models_{\mathcal{D}} \varphi$ for any $\varphi \in \Sigma$.

Definition

For a set $\mathbb{V}$ of valuations and sets $L, R \subseteq \mathcal{L}$ of formulas. We write $L \vdash^{\mathbb{V}}_{\mathcal{D}} R$ if for every $v \in \mathbb{V}$, $v \models_{\mathcal{D}} L$ implies $v \models_{\mathcal{D}} \varphi$ for some $\varphi \in R$. We write $\models_{\mathsf{T}}$ and $\vdash^{\mathbb{V}}_{\mathsf{T}}$ instead of $\models_{\{\mathsf{T}\}}$ and $\vdash^{\mathbb{V}}_{\{\mathsf{T}\}}$.

Counterexample

Consider formula $\neg\Box(p \wedge q) \vee (\Box p \wedge \Box q)$ and valuation:

- $v(p) = v(q) = \mathsf{f}$
- $v(p \wedge q) = \mathsf{F}$
- $v(\Box p) = v(\Box q) = v(\Box p \wedge \Box q) = \mathsf{F}$
- $v(\Box(p \wedge q)) = \mathsf{T}$
- $v(\neg\Box(p \wedge q)) = \mathsf{F}$
- $v(\neg\Box(p \wedge q) \vee (\Box p \wedge \Box q)) = \mathsf{F}$

Level valuations

Definition (Level valuations)

- $\mathbb{V}_{\mathrm{K}}^{\mathcal{F},0} = \{v \mid v \text{ is a } \mathrm{M}_{\mathrm{K}}\text{-legal } \mathcal{F}\text{-valuation}\}$
- $\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m+1} = \left\{ v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F},m} \mid \forall \varphi \in \mathcal{F}.\, v^{-1}[\mathrm{TF}] \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathrm{K}}^{\mathcal{F},m}} \varphi \implies v(\varphi) \in \mathrm{TF} \right\}$
  for $m \geq 0$.

G calculus

$$(\text{WEAK})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (\text{ID})\ \frac{}{\Gamma, \varphi \Rightarrow \varphi, \Delta} \qquad (\text{CUT})\ \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta} \qquad (\supset\Rightarrow)\ \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \supset \psi, \Delta}$$

$$(\Rightarrow\supset)\ \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \supset \psi, \Delta} \qquad (\wedge \Rightarrow)\ \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}$$

$$(\vee \Rightarrow)\ \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

Modal calculi

$$(K)\ \frac{\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi} \qquad (T)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad (4)\ \frac{\Box\Gamma_1, \Gamma_2 \Rightarrow \varphi}{\Box\Gamma_1, \Box\Gamma_2 \Rightarrow \Box\varphi}$$

We call $\mathsf{G}_{\mathtt{K}} = \mathsf{G} + K$, $\mathsf{G}_{\mathtt{K4}} = \mathsf{G} + 4$, $\mathsf{G}_{\mathtt{S4}} = \mathsf{G} + T + 4$

Derivability

Definition

Given a sequent $\Gamma \Rightarrow \Delta$, by $\vdash^{\mathcal{F},m}_{\mathrm{G_{S4}}} \Gamma \Rightarrow \Delta$ we mean that there is a derivation of the sequent $\Gamma \Rightarrow \Delta$ in $\mathrm{G_{S4}}$ with 4-depth of m, where 4-depth of the derivation denotes the maximal number of (4) rule applications used in any branch of the derivation. For an ω-sequent $L \Rightarrow R$ we write $\Vdash^{\mathcal{F},m}_{\mathrm{G_{S4}}} L \Rightarrow R$ if for some finite sub-sequent $\Gamma \Rightarrow \Delta$ of $L \Rightarrow R$, $\vdash^{\mathcal{F},m}_{\mathrm{G_{S4}}} \Gamma \Rightarrow \Delta$.

Results for S4

Nmatrix for S4

Definition

Nmatrix $M_{S4} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, with $\mathcal{V} = \{t, f, T\}$ $\mathcal{D} = \{t, T\}$ and where $\mathcal{O}$ is defined by the following tables:

$x \tilde{\wedge} y$	T	t	f
T	$\mathcal{D}$	$\mathcal{D}$	$\{f\}$
t	$\mathcal{D}$	$\mathcal{D}$	$\{f\}$
f	$\{f\}$	$\{f\}$	$\{f\}$

$x \tilde{\vee} y$	T	t	f
T	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
t	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
f	$\mathcal{D}$	$\mathcal{D}$	$\{f\}$

x	$\tilde{\neg} x$
T	$\{f\}$
t	$\{f\}$
f	$\mathcal{D}$

x	$\tilde{\Box} x$
t	$\{f\}$
f	$\{f\}$
T	$\{T\}$

Level valuations for M_{S4}

Definition (Level valuations)

- $\mathbb{V}_{S4}^{\mathcal{F},0} = \{v \mid v \text{ is a } M_{S4}\text{-legal } \mathcal{F}\text{-valuation}\}$
- $\mathbb{V}_{S4}^{\mathcal{F},m+1} = \left\{ v \in \mathbb{V}_{S4}^{\mathcal{F},m} \mid \forall \varphi \in \mathcal{F}.\, v^{-1}[\mathsf{T}] \vdash_{\mathcal{D}}^{\mathbb{V}_{S4}^{\mathcal{F},m}} \varphi \implies v(\varphi) = \mathsf{T} \right\}$

  for $m > 0$.

Soundness

Lemma (Soundness for (T))

Suppose that $\Delta \cup \Gamma \cup \{\varphi, \Box\varphi\} \subseteq \mathcal{F}$ *and* $\varphi, \Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{S4}}^{\mathcal{F},m}} \Delta$. *Then* $\Box\varphi, \Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{S4}}^{\mathcal{F},m}} \Delta$.

Lemma (Soundness for (4))

Suppose that $\Box\Gamma_1 \cup \Gamma_2 \cup \Box\Gamma_2 \cup \{\varphi, \Box\varphi\} \subseteq \mathcal{F}$ *and* $\Box\Gamma_1, \Gamma_2 \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{S4}}^{\mathcal{F},m-1}} \varphi$. *Then* $\Box\Gamma_1, \Box\Gamma_2 \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{S4}}^{\mathcal{F},m}} \Box\varphi$. *For* $m > 0$.

Completeness

For completeness one should provide a countermodel for an ω-sequence, that is not derivable.

Definition (Maximal and consistent ω-sequent)

Let $\mathcal{F} \subseteq \mathcal{L}$ and $m \geq 0$. A $\mathcal{F}$-ω-sequent $L \Rightarrow R$ is called:

1. $\mathcal{F}$-*maximal* if $\mathcal{F} \subseteq L \cup R$.
2. $\langle \mathsf{G}_{\mathsf{S4}}, \mathcal{F}, m \rangle$-*consistent* if $\nVdash^{\mathcal{F},m}_{\mathsf{G}_{\mathsf{S4}}} L \Rightarrow R$.
3. $\langle \mathsf{G}_{\mathsf{S4}}, \mathcal{F}, m \rangle$-*maximal-consistent* (in short, $\langle \mathsf{G}_{\mathsf{S4}}, \mathcal{F}, m \rangle$-*max-con*) if it is $\mathcal{F}$-maximal and $\langle \mathsf{G}_{\mathsf{S4}}, \mathcal{F}, m \rangle$-consistent.

Countermodel

Definition

We denote $\mathbb{B}_{\mathcal{F}}^{X} = \{\psi \in \mathcal{F} \mid \Box\psi \in X\}$.

Then

$$v(\mathcal{F}, L \Rightarrow R, m) =$$

$m = 0$

$$\lambda\varphi.\begin{cases} \mathrm{T}, & \varphi \in L \wedge \Box\varphi \notin R \\ \mathrm{t}, & \varphi \in L \wedge \Box\varphi \in R \\ \mathrm{f}, & \varphi \in R \end{cases}$$

$m > 0$

$$\lambda\varphi.\begin{cases} \mathrm{T}, & \varphi \in L \wedge \Vdash_{\mathrm{G_{S4}}}^{\mathcal{F}, m-1} \Box\mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi \\ \mathrm{t}, & \varphi \in L \wedge \nVdash_{\mathrm{G_{S4}}}^{\mathcal{F}, m-1} \Box\mathbb{B}_{\mathcal{F}}^{L} \Rightarrow \varphi \\ \mathrm{f}, & \varphi \in R \end{cases}$$

Completeness

Theorem (Completeness)

Let $\mathcal{F} \subseteq \mathcal{L}$ be closed under subformulas. Suppose $L \Rightarrow R$ is a $\langle \mathrm{G}_{\mathtt{S4}}, \mathcal{F}, m+1 \rangle$-max-con ω-sequent, then there is a valuation $v \in \mathbb{V}_{\mathtt{S4}}^{\mathcal{F},m}$, such that $v \not\models_{\mathcal{D}} L \Rightarrow R$.

Completeness lemmas

Lemma

Let $\mathcal{F} \subseteq \mathcal{L}$ be closed under subformulas. Given $L \Rightarrow R$ is a $\langle \mathrm{G}_{\mathtt{S4}}, \mathcal{F}, 0\rangle$-max-con ω-sequent. Then $v(\mathcal{F}, L \Rightarrow R, 0) \in \mathbb{V}_{\mathtt{S4}}^{\mathcal{F},0}$.

Lemma

Let $\mathcal{F} \subseteq \mathcal{L}$ be closed under subformulas. Given $L \Rightarrow R$ is a $\langle \mathrm{G}_{\mathtt{S4}}, \mathcal{F}, m\rangle$-max-con ω-sequent. Then $v(\mathcal{F}, L \Rightarrow R, m) \in \mathbb{V}_{\mathtt{S4}}^{\mathcal{F},0}$.

Lemma

Let $\mathcal{F} \subseteq \mathcal{L}$ be closed under subformulas. Given $L \Rightarrow R$, a $\langle \mathrm{G}_{\mathtt{S4}}, \mathcal{F}, m\rangle$-max-con ω-sequent. If for any $\mathcal{F} - \omega$-sequent $L' \Rightarrow R'$ and for any $k < m$ we have $L' \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathtt{S4}}^{\mathcal{F},k}} R'$ implies $\Vdash_{\mathrm{G}_{\mathtt{S4}}}^{\mathcal{F},k} L' \Rightarrow R'$. Then $v(\mathcal{F}, L \Rightarrow R, m) \in \mathbb{V}_{\mathtt{S4}}^{\mathcal{F},m}$.

Effectiveness

Algorithm

Algorithm 1 Deciding $\Gamma \vdash_{\mathcal{D}}^{\mathbb{V}_{\mathrm{S4}}} \varphi$.

1: $\mathcal{F} \leftarrow sub(\Gamma \cup \{\varphi\})$
2: $m \leftarrow 3^{|\mathcal{F}|}$
3: **for** $v \in \mathbb{V}_{\mathrm{S4}}^{\mathcal{F},m}$ **do**
4:     **if** $v \models_{\mathcal{D}} \Gamma$ and $v \not\models_{\mathcal{D}} \varphi$ **then**
5:         **return** (“NO”, v)
6: **return** “YES”

Lemma

For a finite set $\mathcal{F}$ of formulas, $(\bigcap_{i<\omega} \mathbb{V}_{\mathrm{S4}}^{\mathcal{F},i} =)\mathbb{V}_{\mathrm{S4}}^{\mathcal{F}} = \mathbb{V}_{\mathrm{S4}}^{\mathcal{F},3^{|\mathcal{F}|}}$.

Extension

The algorithm is correct, but in applications we might need a construction of the countermodel. If $\Gamma \not\models_{\mathcal{D}}^{\mathbb{V}_{S4}} \varphi$, then there is a $u \in \mathbb{V}_{S4}$, such that $u \models_{\mathcal{D}} \Gamma$ and $u \not\models_{\mathcal{D}} \varphi$. However, we don't know anything about the connection between u and v from the algorithm output.

This motivates us to investigate the existance of a total extension $v' \in \mathbb{V}_{S4}$, such that $v'|_{\mathcal{F}} = v$.

Kripke models correspondence

Kripke models

Definition

Given $\mathcal{F} \subseteq \mathcal{L}$ is closed under subformula and $v \in \mathbb{V}_{\mathrm{K}}^{\mathcal{F}}$ and a pointed Kripke model $\langle \mathfrak{M}, x \rangle$. We write $\langle \mathfrak{M}, x \rangle \dashv\vdash \langle \mathcal{F}, v \rangle$ if for any $\varphi \in \mathcal{F}$

- $v(\varphi) = \mathsf{t}$ iff. $\mathfrak{M}, x \models \varphi$ and there is an $y \in W$, such that xRy and $\mathfrak{M}, y \not\models \varphi$
- $v(\varphi) = \mathsf{f}$ iff. $\mathfrak{M}, x \not\models \varphi$ and there is an $y \in W$, such that xRy and $\mathfrak{M}, y \not\models \varphi$
- $v(\varphi) = \mathsf{T}$ iff. $\mathfrak{M}, x \models \varphi$ and for any $y \in W$ such that xRy,$\mathfrak{M}, y \models \varphi$
- $v(\varphi) = \mathsf{F}$ iff. $\mathfrak{M}, x \not\models \varphi$ and for any $y \in W$ such that xRy, $\mathfrak{M}, y \models \varphi$

Summary

Summary

- completeness and soundeness were established
- does extension exist?
- how to build a kripke model, corresponding to a valuation?

- Kearns, John T. “Modal Semantics without Possible Worlds”. In: *J. Symb. Log.* 46.1 (1981), pp. 77–86.
- Lahav, Ori and Yoni Zohar. “Effective Semantics for the Modal Logics K and KT via Non-deterministic Matrices”. In: *Automated Reasoning - 11th International Joint Conference, IJCAR 2022, Haifa, Israel, August 8-10, 2022, Proceedings*. Ed. by Jasmin Blanchette, Laura Kovács, and Dirk Pattinson. Vol. 13385. Lecture Notes in Computer Science. Springer, 2022, pp. 468–485. DOI: 10.1007/978-3-031-10769-6_28. URL: https://doi.org/10.1007/978-3-031-10769-6_28.